VIII. *On the Mathematical Theory of Sound.* *By the Rev.* S. Earnshaw, *M.A.*, *Sheffield.* *Communicated by Professor* W. H. Miller, *F.R.S.*

Received November 20, 1858,—Read January 6, 1859*.

In making certain investigations on the properties of the sound-wave, transmitted through a small horizontal tube of uniform bore, I found reason for thinking that the equation

$$\frac{dy}{dt}=\mathrm{F}\left(\frac{dy}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

must always be satisfied; F being a function of a form to be determined. Differentiating this equation with regard to t, we find

$$\frac{d^2y}{dt^2}=\left\{\mathrm{F}'\left(\frac{dy}{dx}\right)\right\}^2\cdot\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

which by means of the arbitrary function F can be made to coincide, not only with the ordinary dynamical equation of sound, but with any dynamical equation in which the ratio of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ can be expressed in terms of $\frac{dy}{dx}$.

Equation (1.) is a partial first integral of (2.), and by means of it we shall be able to obtain a final integral of (2.), which will be shown to be the general integral of (2.) for wave-motion, propagated in one direction only in such a tube as we have supposed, by its satisfying all the conditions of such wave-motion.

It will be convenient to begin with the simplest case of sound,—that in which the development of heat and cold is neglected.

I. Wave-motion when change of temperature is neglected.

1. The equations for this case of motion are, the dynamical equation

$$\left(\frac{dy}{dx}\right)^2\cdot\frac{d^2y}{dt^2}=\mu\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

and the equation of continuity,

$$\frac{dy}{dx}=\frac{\varrho_0}{\varrho}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

ϱ_0, p_0 are the equilibrium density and pressure at any point of the fluid; ϱ, p the same for a particle in motion; x the equilibrium distance of the same particle from a fixed plane cutting the tube at right angles; and t is the time when the same particle, being in motion, is at the distance y from the same plane; μ is the constant which connects ϱ and p by Boyle's law $p=\mu\varrho$.

* Subsequently recast and abridged by the author, but without introducing new matter.

On comparing (3.) with (2.), we find $\frac{dy}{dx}\cdot F'\left(\frac{dy}{dx}\right)=\pm\sqrt{\mu}$; or, for brevity, writing α for $\frac{dy}{dx}$, we have

$$F'(\alpha)=\pm\frac{\sqrt{\mu}}{\alpha},$$

and $$\therefore\quad F(\alpha)=C\pm\sqrt{\mu}\log_{\varepsilon}(\alpha).$$

But as $\frac{dy}{dt}=F(\alpha)$ from (1.), it follows that

$$dy=\frac{dy}{dx}\cdot dx+\frac{dy}{dt}\cdot dt$$
$$=\alpha dx+F(\alpha).dt,$$

which being integrated in the usual manner, substituting at the same time for $F(\alpha)$ its value, gives

$$\left.\begin{array}{l} y=\alpha x+(C\pm\sqrt{\mu}\log_{\varepsilon}\alpha)t+\varphi(\alpha) \\ 0=\alpha x\pm\sqrt{\mu}t+\alpha\varphi'(\alpha) \end{array}\right\} \quad \ldots \quad (5.)$$

Between these equations, if we eliminate α, we have then the integral of equation (3.).

2. From equation (4.) we see that $\alpha=\frac{\varrho_0}{\varrho}$; and if we represent by u the velocity of the particle whose place is y, we find

$$u=\frac{dy}{dt}=C\pm\sqrt{\mu}\log_{\varepsilon}\alpha,$$
$$=C\pm\sqrt{\mu}\log_{\varepsilon}\left(\frac{\varrho_0}{\varrho}\right);$$

and $$\therefore\quad \varrho=\varrho_0\varepsilon^{\mp\frac{u-C}{\sqrt{\mu}}}.$$

3. To determine the arbitrary constant C, we observe that $\varrho=\varrho_0$ and $u=C$ are always simultaneous equations. But the former belongs to the confines of the wave, where in fact $u=0$; and therefore $C=0$. Hence for a wave transmitted through a medium which is itself at rest beyond the limits of the wave, we have these equations*:—

$$\varrho=\varrho_0\varepsilon^{\mp\frac{u}{\sqrt{\mu}}}. \quad \ldots \quad (6.)$$

$$\left.\begin{array}{l} y=\alpha x\pm\sqrt{\mu}\log_{\varepsilon}\alpha.t+\varphi(\alpha) \\ 0=\alpha x\pm\sqrt{\mu}t+\alpha\varphi'(\alpha) \end{array}\right\} \quad \ldots \quad (7.)$$

* If x and α be eliminated between the equations (7.) and $u=\pm\sqrt{\mu}\log_{\varepsilon}\alpha$, we shall obtain the equation

$$u=f\{y-(u\mp\sqrt{\mu})t\},$$

which was first obtained, though in a very different manner from that employed in this paper, by M. Poisson, and printed in the Journal of the Polytechnique School, tome vii. It seems not to have occurred to him, however, that by means of this equation he might effect another integration of the equations of fluid motion, and thus discover the relation between ρ and u, whereby his solution would have been completed.

Several of the properties of wave-motion, depending on the gradual change of type, which are included in this equation of M. Poisson's, were first brought forward and discussed by Professor Stokes in the Philosophical Magazine for November 1848, and by the Astronomer Royal in June 1849. In the latter

4. We have now to express these results in terms of the original genesis of the motion. Let us suppose the motion generated by a piston pushed forwards in the tube in a given manner. Let the piston at the time T (having the same origin as t) be at the distance Y from the plane of reference, and moving forwards with the velocity U; and by R denote the density of the air in contact with the piston at that moment. For all particles in contact with the piston $x=0$ (we suppose the piston to commence its motion at the origin of x). Then since at the time T the particles in contact with the piston are within the limits of the wave, equations (6.) and (7.) must be satisfied;

$$\therefore \quad \mathrm{R}=\rho_0 \varepsilon^{\mp\frac{\mathrm{U}}{\sqrt{\mu}}}$$

$$\left.\begin{aligned} \mathrm{Y} &= \pm\sqrt{\mu}\log_\varepsilon \alpha' . \mathrm{T}+\varphi(\alpha') \\ 0 &= \pm\sqrt{\mu}\mathrm{T}+\alpha'\varphi'(\alpha') \end{aligned}\right\} \quad \text{(8.)}$$

In these equations $\alpha'=\frac{\rho_0}{\mathrm{R}}$, and at present we have not sufficiently connected the two systems of equations (7.) and (8.). We shall further connect them by assuming $\mathrm{R}=\rho$, which gives $\alpha'=\alpha$; the effect of which assumption is to limit the meaning of T, Y, U as follows:—

T is the time of genesis of the density ρ which at the time t has been transmitted to the place denoted by y;

Y is the place where the density ρ was generated;

U is the velocity of the piston when ρ was generated by it.

We may now write α for α', and then eliminate α, $\varphi(\alpha)$, and $\varphi'(\alpha)$ between the four equations (7.), (8.). By this means we obtain

$$y=\mathrm{Y}+(\mathrm{U}\mp\sqrt{\mu})(t-\mathrm{T}). \quad \text{(9.)}$$

$$x=\mp\sqrt{\mu}\varepsilon^{\mp\frac{\mathrm{U}}{\sqrt{\mu}}}(t-\mathrm{T}). \quad \text{(10.)}$$

$$\rho=\rho_0\varepsilon^{\mp\frac{\mathrm{U}}{\sqrt{\mu}}}=\rho_0\varepsilon^{\mp\frac{u}{\sqrt{\mu}}}. \quad \text{(11.)}$$

5. By these equations the state of a wave at any moment is connected with its genesis; and they contain in fact the complete solution of the problem of every kind of motion, in a tube, which can be generated by a piston.

6. From (11.) it appears that $u=\mathrm{U}$; that is, that the particle-velocity generated by the piston is transmitted through the medium without suffering any alteration. The same equation (11.) shows that between the density and the velocity there is an invariable relation, which is independent of the law of original genesis of the motion; so that in the same wave, or in different waves, wherever there is the same density, there will also be the same velocity.

7. One of the most obvious facts on looking at the equations just found is, that for the same genesis there are *two* values of x, two of y, and two of ρ. The signification of

Number of the Magazine it also appears that Professor De Morgan had discovered and communicated to the Astronomer Royal two particular forms of the function F; without perceiving, however, that a slight generalization of his results would put him in the way to the integral expressed by the equations (5.).

this is, that a single disturbance generates *two* waves; and (11.) shows that for one of them ϱ is *greater*, and for the other *less* than ϱ_0. Equation (10.) shows that they are propagated in contrary directions on opposite sides of the piston, and are therefore not parts of the *same* wave.

8. In the genesis of the wave we have supposed the piston pushed forwards, that is, in the direction of $+x$. Hence for the wave generated on that side of the piston we must, as appears from (10.), take the lower sign, which in (11.) gives ϱ greater than ϱ_0. This wave we call the *positive* wave, and the wave of *condensation.* For the wave generated on the other side of the piston we must take the upper sign, which gives ϱ less than ϱ_0; and this wave we call the *negative* wave, and the wave of *rarefaction.*

9. As it will be useful to have a definition of these two waves, which shall be independent of their position with regard to the generating piston, we may state that in general,—

a *positive* wave is one in which the motions of the particles are in the direction of wave-transmission: and

a *negative* wave is one in which the motions of the particles are in a direction opposite to that of wave transmission.

10. If ϱ_1 and ϱ_2 be the densities of the air in contact with the piston before and behind at any moment, and if p_1 and p_2 be the corresponding pressures; then from (11.) we have

$$\varrho_1=\varrho_0\varepsilon^{\frac{\mathrm{U}}{\sqrt{\mu}}}, \text{ and } \varrho_2=\varrho_0\varepsilon^{-\frac{\mathrm{U}}{\sqrt{\mu}}},$$

and $$\therefore \quad \varrho_1\varrho_2=\varrho_0^2;$$

which may be thus expressed in words:—if a piston move in a tube, filled with air, in any manner whatever, the densities of the air in contact with it at its front and back are such that the equilibrium density is a mean proportional between them. And since $p=\mu\varrho$, we have $p_1p_2=p_0^2$, which furnishes us with a similar property for the *pressures* on the piston.

11. Since $p_1=p_0\varepsilon^{\frac{\mathrm{U}}{\sqrt{\mu}}}$ and $p_2=p_0\varepsilon^{-\frac{\mathrm{U}}{\sqrt{\mu}}}$, it follows that the resistance to the motion of the piston (calling S its area) is

$$(p_1-p_2)\mathrm{S}=\left(\varepsilon^{\frac{\mathrm{U}}{\sqrt{\mu}}}-\varepsilon^{-\frac{\mathrm{U}}{\sqrt{\mu}}}\right)p_0\mathrm{S}.$$

Hence in different gases, if p_0 be the same in all, those will offer the greatest resistance to the piston for which μ is the least.

It will be convenient from this point to consider the two kinds of waves separately.

1. *The Wave of Condensation.*

12. The equations for this wave are

$$y=\mathrm{Y}+(\sqrt{\mu}+\mathrm{U})(t-\mathrm{T})$$

$$x=\sqrt{\mu}\varepsilon^{\frac{\mathrm{U}}{\sqrt{\mu}}}(t-\mathrm{T})$$

$$\varrho=\varrho_0\varepsilon^{\frac{\mathrm{U}}{\sqrt{\mu}}}.$$

13. Now with respect to the genesis of this wave, we have seen that U must satisfy the same conditions as u, and Y as y. But $u=\frac{dy}{dt}$, therefore $\mathrm{U}=\frac{d\mathrm{Y}}{d\mathrm{T}}$: and again, as one of the equations of the general integral (7.) was obtained from the other by differentiation with regard to α, it follows that both α and U must vary continuously; and that $\frac{d\mathrm{U}}{d\mathrm{T}}$ must not pass through infinity; in other words, if the velocity of the piston vary it must vary continuously. Neither Y nor U must be discontinuous with regard to T. Hence there must be no discontinuity of pressure within the limits of the wave at its genesis: and if discontinuity should afterwards occur in the wave during its transmission, our equations will cease to be applicable for that part of the wave where the discontinuity has occurred. For the wave in any one position may be supposed to generate its next position; and a piston or diaphragm may at any time be supposed to act the part of the generating wave. What is necessary for the diaphragm to observe as a law of genesis must be necessary for the wave considered as the generator of its next position; and therefore the part of the wave (if any) where discontinuity occurs will be beyond the reach of our equations.

14. It has been shown that the density ϱ, which at the time t is at the distance y from the plane of reference, was generated at the time T when its distance from the same plane was Y. Hence it has been transmitted through the space $y-\mathrm{Y}$ in the time $t-\mathrm{T}$, and consequently the velocity of its transmission (as appears from the first equation of (12.)) is

$$\sqrt{\mu}+\mathrm{U}.$$

15. The wave as a whole is included between two points of it for each of which $\mathrm{U}=0$, and consequently for each of those points the velocity of transmission is $\sqrt{\mu}$. Hence the wave as a whole is transmitted with this uniform velocity. But all the parts of the wave, with the exception of its front and rear, are transmitted with velocities greater than this,—with velocities dependent on their respective densities. Hence every part of the wave, with the exception of its rear, is perpetually gaining on the front, and the result is a *constant change of type*,—the more condensed parts hurrying towards the front, with velocities greater as their densities are greater. This cannot go on perpetually without its happening at length that a *bore* (or tendency to a discontinuity of pressure) will be formed in front; which will force its way, in violation of our equations, faster than at the rate of $\sqrt{\mu}$ feet per second; and consequently in experiments, made on sound at long distances from the origin of the sound-wave, we should expect the actual velocity observed to be greater than $\sqrt{\mu}$, especially if the sound be a violent one, generated with extreme force (see art. 17).

16. We have seen that the velocity of transmission of the density ϱ is $\sqrt{\mu}+\mathrm{U}$. Now the velocity of the particles where the density is ϱ is u, which we have shown to be equal to U. In a certain sense we may consider the velocity u to be a wind-velocity in that part of the medium, and then we have an indefinitely small disturbance at that point transmitted in that wind with the velocity $\sqrt{\mu}$ imposed upon the wind. In other

words, transmission-velocity is superimposed on particle-velocity, and in this sense transmission-velocity is everywhere the same, and equal to $\sqrt{\mu}$. A wave passes by every particle with this velocity, whatever be the particular and varying density of the medium where the particle is situated.

17. Since a wave's front cannot move faster than with the velocity $\sqrt{\mu}$, if the generating piston move faster than with this velocity, it will generate a bore; and from this we infer that a bore always moves with a velocity greater than $\sqrt{\mu}$; for wherever a bore may be situated at *any time*, we may suppose it to be *just* generated by a piston. If we write $\sqrt{\mu}$ for U, we find $p=\mu\rho=\mu\rho_0\varepsilon=\varepsilon p_0$. Consequently if the piston press upon the resisting air with a pressure exceeding ε atmospheres, a bore will be instantly formed.

I have defined a bore to be *a tendency to discontinuity of pressure*; and it has been shown that as a wave progresses such a tendency necessarily arises. As, however, discontinuity of pressure is a physical impossibility, it is certain Nature has a way of avoiding its actual occurrence. To examine in what way she does this, let us suppose a discontinuity to have actually occurred at the point A, in a wave which is moving forwards. Imagine a film of fluid at A forming a section at right angles to the tube. Then on the back of this film there is a certain pressure which is discontinuous with respect to the pressure on its front. To restore continuity of pressure, the film at A will rush forward with a *sudden* increase of velocity, the pressure in the front of the film not being sufficient to preserve continuity of velocity. In so doing the film will play the part of a piston generating a bit of wave in front, and a small regressive wave behind. The result will be a prolongation of the wave's front, thereby increasing the original length of the wave, and producing simultaneously a feeble regressive wave of a negative character.

Now all this supposes the discontinuity to have actually occurred, which, as has been said, is a physical impossibility. For actual occurrence we must therefore substitute a tendency to occur, and modify the preceding reasoning thus:—

Nature so contrives, that as the discontinuity is in its initial stage of beginning to take place, its actual occurrence is prevented by a *gradual* (not *sudden*) prolongation of the wave's front, and by the *constant* casting off, from its front in a retrogressive direction, of a long continuous wave of a negative character, which will be of greater or less intensity according as the tendency to discontinuity is more or less intense in the original wave.

The casting off of this long wave will probably manifest itself audibly as a continuous *hiss* or *rushing* sound.

Hence a sound-wave, from the moment that a tendency to discontinuity begins in its front, has the property of constantly prolonging itself in front, and by this means its front travels faster than at the rate $\sqrt{\mu}$. Those sounds also will travel most rapidly whose genesis was most violent; and gentlest sounds travel with velocities not much differing from $\sqrt{\mu}$. I should expect, therefore, that in circumstances where the human voice can be heard at a sufficiently great distance, the *command* to fire a gun, if instantly obeyed, and the *report* of the gun, might be heard at a long distance in an inverse order; i. e. *first* the report of the gun, and *then* the word "fire*." In a slight degree, therefore, the experimental velocity of sound will depend on its intensity, and the violence of its genesis. I consider this article as tending to account for the *discrepancy* between the *calculated* and *observed* velocities of sound (which most experimentalists have remarked and wondered at), when allowance is made (as will be done in a future part of this paper) for change of temperature.

18. It seems reasonable to suppose that the audible character of a wave is in some way dependent on its type; and consequently, if this be the case, the sound undergoes a perpetual modification as the distance of transmission increases. One modification of the sound-wave is, as we have seen, the formation of a bore in front; but there is another which cannot but have some influence on its audible properties, as it corresponds to a remarkable change of type; and this takes place when the greater densities begin to overtake the less.

Now when one degree of density overtakes another, the values of y corresponding to those two densities are equal; and hence at the time t the equation

$$y=\mathrm{Y}+(\sqrt{\mu}+\mathrm{U})(t-\mathrm{T}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12.)$$

will give two equal values of y for two consecutive values of T. Hence differentiating it with regard to T, remembering that t is constant, or the same for both, as is also y, we have

$$0=\mathrm{U}-(\sqrt{\mu}+\mathrm{U})+(t-\mathrm{T})\frac{d\mathrm{U}}{d\mathrm{T}},$$

or

$$t=\mathrm{T}+\frac{\mu}{\frac{d\mathrm{U}}{d\mathrm{T}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13.)$$

The right-hand member of this equation is of course a continuous expression, and therefore its least or minimum value will be the value of t when the modification of type, of which we are speaking, *first begins* to take place; and because of the continuity of (13.), this modification once begun will gradually spread itself over the fore-part of the wave. Now t will be a minimum when

$$\left(\frac{d\mathrm{U}}{d\mathrm{T}}\right)^2=\mu\cdot\frac{d^2\mathrm{U}}{d\mathrm{T}^2}.$$

From this equation we may find T, the time of genesis of that part of the wave where this modification begins. Then (13.) will give t, the actual time when the modification begins; and (12.) will give the place in the tube where it begins.

19. It is perhaps impossible to say what is the audible characteristic corresponding to the wave-modification just investigated; but whatever it be, we perceive from (13.) that those sound-waves soonest begin to be affected by it for which $\frac{d\mathrm{U}}{d\mathrm{T}}$ is largest; *i. e.* those

* See Supplement to Appendix of PARRY's Voyage in 1819–20, Art. "Abstract of Experiments to determine the Velocity of Sound."

whose genesis is most violent. And we may also consider it as proved that those sounds will retain their original characteristics the longest which are the most gently generated.

It is also quite evident from (13.) if the same cause generate sound-waves in different tubes filled with different gases, the wave will be soonest affected by the above modification in that tube which contains the gas for which μ is least.

We come now to speak of

2. *The Wave of Rarefaction.*

20. We shall obtain the equations for this kind of wave by writing $-\mathrm{U}$ for $+\mathrm{U}$ in the equations of art. 12, which is equivalent to supposing a negative wave generated on the $+y$ side of the piston. Hence the equations of a negative wave are

$$y=\mathrm{Y}+(\sqrt{\mu}-\mathrm{U})(t-\mathrm{T}),$$

$$x=\sqrt{\mu}\varepsilon^{-\frac{\mathrm{U}}{\sqrt{\mu}}}(t-\mathrm{T}),$$

$$\rho=\rho_0\varepsilon^{-\frac{\mathrm{U}}{\sqrt{\mu}}}.$$

21. Reasoning in the same manner as in art. 14, it appears that the velocity with which the density ρ is transmitted is

$$\sqrt{\mu}-\mathrm{U}.$$

From this it appears that, speaking generally, the velocity of transmission of every part of a negative wave is less than of every part of a positive wave. The exceptions to this statement are the front and rear, which in both kinds of waves move with the same velocity $\sqrt{\mu}$, because for those points $\mathrm{U}=0$. It is evident also that the most rarefied parts of a wave will be transmitted the most slowly, and will consequently drop continually towards the rear. Hence in this species of wave, as in the former, a constant change of type takes place; and in the end also a *negative or rarefied bore* will be formed in the *rear* of the wave.

By a process of reasoning analogous to that of art. 17, we infer that a negative sound-wave, from the moment that a tendency to discontinuity begins in its rear, has the property of constantly shortening its rear, and by this means its rear travels faster than at the rate $\sqrt{\mu}$; and also as it progresses it is constantly casting off from its rear in a regressive direction a long continuous wave of a negative character. Art. 18 also admits of easy modification to this kind of wave.

22. The velocity of transmission of a negative wave being $\sqrt{\mu}-\mathrm{U}$, and the last term of this expression admitting of arbitrary increase, it is evident that $\mathrm{U}=\sqrt{\mu}$ is a critical value, and that the part of the wave corresponding to that value of U is stationary. The corresponding value of ρ is $\frac{\rho_0}{\varepsilon}$.

Every part of the wave where the density exceeds this travels forwards; but the parts where the density is less than this are regressive; hence a wave, as a whole, in which ρ

begins at ϱ_0, and after twice passing through $\frac{\varrho_0}{\varepsilon}$ ends at ϱ_0, will have two stationary points in its type, viz. those where $\varrho=\frac{\varrho_0}{\varepsilon}$. Between these points the wave will be stationary though constantly changing type; beyond them progressive.

23. But instead of supposing the piston to generate such a wave as this, let us suppose it to begin from the velocity zero, and according to any proposed law (continuous of course) increase its velocity till it becomes infinite; and let us consider the state of the medium at this moment.

Denote by A and B the places of the piston where its velocity became respectively $\sqrt{\mu}$ and infinite. Then whatever was the law of motion from A to B, and whether AB be great or small, provided it remains of finite length, the density at A will remain unchanged and equal to $\frac{\varrho_0}{\varepsilon}$, and the velocity of every particle as it passes by A will be equal to $\sqrt{\mu}$. The mass of air also which will rush through the section of the tube at A will be $\frac{S\varrho_0\sqrt{\mu}}{\varepsilon}$; and this, be it observed, cannot be made either more or less by causing the piston to move in a different manner from A to B. It is also equally independent of the law of the piston's motion before it reached A. Hence the mass of air that flows through the section at A is altogether independent of the law of the piston's motion throughout its whole course.

24. Now let us inquire what quantity of air rushes through any other section of the tube. In every part where there is motion the same relation between density and velocity obtains, viz. $\varrho=\varrho_0\varepsilon^{-\frac{u}{\sqrt{\mu}}}$; and consequently the quantity which rushes through any section is at the rate of

$$S\varrho_0 u\varepsilon^{-\frac{u}{\sqrt{\mu}}} \text{ per second.}$$

It is obvious this admits of a maximum value, which in the usual manner we find to be

$$\frac{S\varrho_0\sqrt{\mu}}{\varepsilon},$$

at which value $u=\sqrt{\mu}$ and $\varrho=\frac{\varrho_0}{\varepsilon}$.

25. Hence one part of the tube cannot supply air to another part faster than at this rate; and consequently the greatest possible mass of air passes through the section at A: and it may be stated as a general property of motion through a tube, that a gas cannot be conveyed through a tube faster than at the rate of $\frac{S\sqrt{\mu}}{\varepsilon}$ cubic feet per second of gas of the density ϱ_0.

Hence the escaping powers of different gases through equal tubes are proportional to the velocities with which they respectively transmit sound.

26. Since this result is independent of the law of velocity of the air, both before and after passing the section A, we are entitled to say that air cannot rush through a pipe of finite length, even into a vacuum, faster than at the rate of $\frac{S\sqrt{\mu}}{\varepsilon}$ cubic feet per

second. The length of the pipe seems to be a matter of perfect indifference, and may be nothing more than a hole through a partition of finite thickness.

27. Since one part of a tube cannot supply air to, nor convey air away from, another part, A, faster than at the maximum rate, it is easy to see that if the pipe be supposed of finite length, which conveys air into a vacuum, the velocity in every part of the pipe will soon be the same throughout, and equal to $\sqrt{\mu}$, and density everywhere equal to $\frac{\varrho_0}{\varepsilon}$.

From this it would appear that the rate of discharge into a vacuum, which has generally been supposed to be that which is due to the height of the homogeneous atmosphere, is in reality that which is due to the $\left(\frac{1}{2\varepsilon^2}\right)$th part only; that is, to little more than the fifteenth part of it; but this requires correction for change of temperature.

28. If the generating piston move forward and then backward, so as to generate a positive wave followed continuously by a negative wave, they will not separate; for, as we have seen, they are each transmitted, as wholes, at the same rate $\sqrt{\mu}$. But the main body of the positive wave will gradually advance in the type towards its front, and that of the negative wave fall back towards its rear; and consequently for the purposes of audibility the central part of the compound wave, between the front of the positive and the rear of the negative wave, will become so attenuated that it may be considered of little audible effect, after the waves have been in existence a sufficient length of time to allow the formation of bores. The compound wave will therefore have a tendency to produce the audible effect of two separate waves, separated by an interval of space nearly equal to its whole length. If therefore the length of such a compound wave be sufficiently great, it will ultimately produce *two* distinct sounds separated by a very brief interval of time.

29. If the generating piston move backward and then forward, so as to generate a negative wave followed continuously by a positive wave, the positive and negative bores will destroy each other as rapidly as they are formed. This, however, supposes the positive and negative portions of the original compound wave to be equal. If one exceed the other in quantity of motion, the result will be a little modified. A compound wave of the kind supposed in this article will therefore be entirely devoid of bores, and the sound corresponding to it will be free from that harshness which is probably the audible character of a bore.

30. If there be a continuous succession of positive and negative waves, constituting one long compound wave, such a wave will produce a continuous even sound, called a musical note, probably owing its sweetness in some degree to the property just mentioned; and as every negative portion is succeeded by a positive portion, and every positive by a negative, the length of each portion will remain unchangeable, whatever be the distance through which the compound wave travels. Hence the pitch of a musical note cannot change by distance of transmission.

31. Suppose a portion of the tube to be filled with air of a different kind from that which fills the first part. Let $p'_0\ \varrho'_0\ \mu'$ be the quantities for this air which correspond to $p_0\ \varrho_0\ \mu$ of the former; and to prevent the two airs or gases from mixing, let them be supposed to be separated by an impenetrable film without weight and inertia. Then as there is equilibrium in the tube before the wave is generated, we have

$$p_0=p'_0.$$

Let now a wave be generated in the first medium and transmitted towards the second; then when it has reached the common boundary of the media, the velocities of the particles in contact with the film on both sides will always be equal. Let U′ be this velocity at any moment, and U the velocity which the film would have had at that moment, if the second medium had been the same as the first. Then U−U′ is the velocity lost by the particles of the first medium by the resistance due to their contact with the film. In other words, this velocity has been impressed on the particles of the first medium by the resistance of the film, in the reflex direction. This gives rise to a reflex wave in the first medium, which we may consider superimposed on the wind of the original wave. And consequently if p be the pressure at the film due to the original wave, the pressure when this reflex wave has been superimposed, *i. e.* the actual pressure at the film, is $=p\varepsilon^{\frac{U-U'}{\sqrt{\mu}}}$, which $=p_0\varepsilon^{\frac{2U-U'}{\sqrt{\mu}}}$, $\because p=p_0\varepsilon^{\frac{U}{\sqrt{\mu}}}$.

But if we now turn to the other side of the film, the velocity U′ has been impressed upon the particles of the second medium in contact with the film; and hence the pressure of those particles on the film

$$=p'_0\varepsilon^{\frac{U'}{\sqrt{\mu'}}}=p_0\varepsilon^{\frac{U'}{\sqrt{\mu'}}};$$

and consequently, as the pressures on the two sides are equal, we have

$$\frac{U'}{\sqrt{\mu'}}=\frac{2U-U'}{\sqrt{\mu}}.$$

Hence the velocities of the particles at the film, for the *incident*, *reflected* and *refracted* waves, are respectively proportional to

$$\sqrt{\mu}+\sqrt{\mu'},\ \sqrt{\mu}-\sqrt{\mu'},\ \text{and } 2\sqrt{\mu'}.$$

There is nothing new in these formulæ, except that they are here deduced without supposing the motions small.

II. WAVE MOTION WHEN CHANGE OF TEMPERATURE IS NOT NEGLECTED.

32. The heat developed by that change of temperature which is produced by the sudden alteration of density due to the passage of a wave, is probably taken account of by using the following equation as that which connects pressure and density,

$$\frac{p}{p_0}=\left(\frac{\varrho}{\varrho_0}\right)^k;$$

k being the ratio of the specific heat of the gas under a constant pressure, to its specific heat under a constant volume. The dynamical equation takes for this case the following form to be used instead of that in art. 1,

$$\left(\frac{dy}{dx}\right)^{k+1}\cdot\frac{d^2y}{dt^2}=k\mu\cdot\frac{d^2y}{dx^2}.$$

This equation being integrated as explained in art. 1, gives

$$\left.\begin{array}{l} y=\alpha x+\left(\mathrm{C}\mp\frac{2\sqrt{k\mu}}{k-1}\alpha^{-\frac{k-1}{2}}\right)t+\varphi(\alpha) \\ 0=\alpha x\pm\sqrt{k\mu}\alpha^{-\frac{k-1}{2}}t+\alpha\varphi'(\alpha) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

33. From these we obtain

$$u=\frac{dy}{}=\mathrm{C}\mp\frac{2\sqrt{k\mu}}{k-1}\alpha^{-\frac{k-1}{2}},$$

$$=\mathrm{C}\mp\frac{2\sqrt{k\mu}}{k-1}\left(\frac{}{\varrho_0}\right)^{\frac{k-1}{2}}.$$

For the same reasons as before we shall suppose $u=0$ and $\varrho=\varrho_0$ to be simultaneous equations; which gives

$$\mathrm{C}=\pm\frac{2\sqrt{k\mu}}{k-1},$$

and

$$\therefore \quad \left(\frac{\varrho}{\varrho_0}\right)^{\frac{k-1}{2}}=1\mp\frac{(k-1)u}{2\sqrt{k\mu}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

This equation gives the relation between density and velocity; from which that between pressure and velocity is easily found.

34. The general integral (14.) may be expressed in terms of the original genesis precisely in the same manner as was employed in art. 4; and the result is

$$y=\mathrm{Y}+\left(\frac{k+1}{2}\mathrm{U}\mp\sqrt{k\mu}\right)(t-\mathrm{T}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

$$x=\mp\sqrt{k\mu}\left(1\mp\frac{k-1}{2\sqrt{k\mu}}\mathrm{U}\right)^{\frac{k+1}{k-1}}(t-\mathrm{T}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

$$u=\mathrm{U}, \text{ and } p=p_0\left(\frac{\varrho}{\varrho_0}\right)^k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

These equations, with (15.), are those from which the properties of the motion are to be deduced. The degree of modification of former results required by these formulæ will be in most cases sufficiently evident, and need not therefore to be particularly pointed out.

35. The result of art. 10 takes the following form—

$$\varrho_1^{\frac{k-1}{2}}+\varrho_2^{\frac{k-1}{2}}=2\varrho_0^{\frac{k-1}{2}};$$

and that of art. 11 the following—

$$(p_1-p_2)\mathrm{S}=\mathrm{S}p_0\left\{\left(1+\frac{k-1}{2\sqrt{k\mu}}\mathrm{U}\right)^{\frac{2k}{k-1}}-\left(1-\frac{k-1}{2\sqrt{k\mu}}\mathrm{U}\right)^{\frac{2k}{k-1}}\right\}.$$

36. From (16.) it appears that the velocity of transmission of the front and rear of either a positive or negative wave is $\sqrt{k\mu}$; but the velocity of transmission of that part of the wave of which the density is ϱ, is, for a positive wave,

$$\sqrt{k\mu}+\frac{k+1}{2}\mathrm{U};$$

and for a negative wave,

$$\sqrt{k\mu}-\frac{k+1}{2}\mathrm{U}.$$

The part of these expressions to which the bore is due is the term $\frac{k+1}{2}\mathrm{U}$; and as k is known to be greater than unity, this is greater than U; and consequently change of temperature hastens the formation of a bore, and also renders the property of art. 16 inapplicable here.

37. As in the case of a negative wave the equation (15.) involves a negative term, it is manifestly possible for the piston, in generating a negative wave, to move so quickly as to leave a vacuum behind it. The least velocity with which this can happen is

$$\frac{2\sqrt{k\mu}}{k-1},$$

which for common air is about 5722 feet per second. But it is necessary to notice, that in this and similar extreme results, we are hardly justified in supposing k to be constant up to such high velocities.

38. The expression $\mathrm{S}\varrho u$ is a maximum (see art. 24) when

$$u=\frac{2\sqrt{k\mu}}{k+1},$$

which in the case of common air is equal to about 904 feet per second; and the corresponding density is

$$\varrho=\left(\frac{2}{k+1}\right)^{\frac{2}{k-1}}\varrho_0;$$

or, for common air, about $\frac{2}{5}\varrho_0$.

Hence no gas can rush through a pipe faster than at the rate of

$$\sqrt{k\mu}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\mathrm{S}$$

cubic feet per second.

39. The change of temperature due to the transmission of a wave through an elastic medium has been taken account of, by assuming a law different from that of Boyle, to connect pressure with density (art. 32).

If we generalize the law by assuming

$$p=\varphi\left(\frac{\varrho_0}{\varrho}\right),$$

the dynamical equation takes the form

$$\frac{d^2y}{dt^2}=-\frac{1}{\rho_0}\cdot\varphi'\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2}.$$

If now we assume

$$(\mathrm{F}'\alpha)^2=-\frac{\varphi'\alpha}{\rho_0},$$

the integral of the dynamical equation will be

$$\begin{cases} y=\alpha x+(\mathrm{C}\pm\mathrm{F}\alpha)t+f\alpha, \\ x=\mp\mathrm{F}'\alpha.t-f'\alpha, \end{cases}$$

with $\alpha=\frac{\rho_0}{\rho}$, and $u=\mathrm{C}\pm\mathrm{F}\alpha=\mathrm{C}\pm\mathrm{F}\left(\frac{\rho_0}{\rho}\right)$.

40. These equations are true of any motion which can be generated by a piston moving subject to the laws of continuity. See art. 13. The last shows that the relation between velocity and density is independent of the law of genesis of the motion. The medium may be, as a whole, in motion with the uniform velocity $\mathrm{C}\pm\mathrm{F}(1)$, and the motion of the particles caused by the motion of a piston will be superimposed on this. For convenience we shall suppose the medium as a whole at rest, and $\therefore$ $\mathrm{C}\pm\mathrm{F}(1)=0$.

If there be a point, or any number of points, within that part of the medium which is in motion for which $\rho=\rho_0$, for all such points $\alpha=1$, and the equation

$$x=\mp\mathrm{F}'(1).t-f'(1),$$

which is always true for all such points, shows that at those points x changes its value at the rate of $\mathrm{F}'(1)$ feet per second, *i. e.* the front of the wave travels at the rate of

$$\left\{-\frac{\varphi'(1)}{\rho_0}\right\}^{\frac{1}{2}} \text{ feet per second,}$$

which is constant, and depends not at all on the law of genesis, but only on the assumed relation between pressure and density, and not on the *general* value of even that, but only on its limiting value when $\rho=\rho_0$. Now many different forms of the function φ may give the same limiting value; and consequently all the media corresponding to these various forms of φ will transmit a wave, as a whole, with the same velocity. Hence if the relation between pressure and density be given, the wave-velocity may be instantly deduced from the expression $\left\{-\frac{\varphi'(1)}{\rho_0}\right\}^{\frac{1}{2}}$, or from its equal,

$$\left\{\frac{dp}{d\rho}\right\}_0^{\frac{1}{2}};$$

using the subscript 0 to signify that after the differentiation has been performed ρ_0 is to be written for ρ.

41. Since $u=\mathrm{C}\pm\mathrm{F}\left(\frac{\rho_0}{\rho}\right)$, by differentiation we obtain

$$\frac{du}{d\rho}=\mp\frac{1}{\rho}\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}}.$$

42. And if c denote the velocity of transmission of the density ϱ, then we have

$$c=\mathrm{C}\pm\mathrm{F}\left(\frac{\varrho_0}{\varrho}\right)\mp\frac{\varrho_0}{\varrho}\mathrm{F}'\left(\frac{\varrho_0}{\varrho}\right)=u\mp\left(\frac{dp}{d\varrho}\right)^{\frac{1}{2}};$$

consequently

$$\frac{dc}{d\varrho}=\mp\frac{1}{\varrho}\cdot\frac{d}{d\varrho}\left(\varrho^2\frac{dp}{d\varrho}\right)^{\frac{1}{2}}.$$

Now the former of these equations shows that unless the term $\left(\frac{dp}{d\varrho}\right)^{\frac{1}{2}}$ be constant, the property of the superposition of wave-transmission on particle-velocity, proved in art. 16, does not hold good. But if it be constant, then $p=\mu\varrho+\mu'$; which is the general relation between pressure and density when that principle of superposition holds good. Hence, as mentioned in art. 36, the development of heat puts an end to this property in all known gases.

43. In the case of negative waves we may institute a method of reasoning similar to that employed in arts. 23 *et seq.*, and arrive at analogous results. We shall also find that, taking $c=\left(\frac{dp}{d\varrho}\right)^{\frac{1}{2}}-u$ for this case, the maximum value of $\mathrm{S}\varrho u$ will occur in that section of the tube where $c=0$; from which it follows that at that section

$$u=\left(\frac{dp}{d\varrho}\right)^{\frac{1}{2}};$$

which is always possible and finite. Hence may be determined the limit to the quantity of a gas that can pass through a pipe in a given time, even into a vacuum.

44. The expression in art. 42 for $\frac{dc}{d\varrho}$ shows that c is in general a function of ϱ, so that in general there will be a constant change of type. In one case, however, there will be no change of type. This will take place when $\frac{dc}{d\varrho}=0$, that is when $\varrho^2.\frac{dp}{d\varrho}$ is constant. Assume for this case

$$\varrho^2\frac{dp}{d\varrho}=\mathrm{B};$$

$$\therefore p=\mathrm{A}-\frac{\mathrm{B}}{\varrho}.$$

This equation expresses the nature of the medium which is distinguished by the property, that it transmits waves without change of type. And if we pass from this to the dynamical equation, we find

$$\frac{d^2y}{dt^2}=\frac{\mathrm{B}}{\varrho_0^2}\cdot\frac{d^2y}{dx^2}.$$

Now it has been usual to reduce the equation (3.) to this form for the purposes of approximation; but the process appears to be allowable only so far as the equation $p=\mathrm{A}-\frac{\mathrm{B}}{\varrho}$ may be taken to be a physical approximation to BOYLE'S law $p=\mu\varrho$. To me it does not seem to be an allowable approximation; and consequently I do not consider the solution of the dynamical equation, which has been obtained by this means, to be applicable to the problem of sound at all. Many *analytically* approximate forms might be invented and used for BOYLE's law, and each would have its peculiar physical attributive effects on the sound-wave; and we might thus, by adopting first one and then another of these analytically approximate laws, invent *ad libitum* an inexhaustible list of properties of the sound-wave which have no real existence where BOYLE's law is strictly true. From which therefore it would seem to be a necessary consequence, that an equation between p and ϱ must not only be *analytically* but also *physically* approximative, in order that the results deduced from it may be accepted as real approximations to the true laws of nature.

45. By means of the expressions for $\frac{du}{d\varrho}$ and $\frac{dc}{d\varrho}$, we may not only discover the properties of motion in a tube without having recourse to the usual equations, when the relation between ϱ and p is known, but we may also solve many inverse problems.

Also, if the tube be filled with a medium of such a nature that the relation between p and ϱ changes continuously from point to point, or is different in different parts, yet if $\left(\frac{dp}{d\varrho}\right)_0$ has the same value everywhere, waves will travel through the tube with a uniform velocity.

If the nature of the medium should vary slowly and continuously, the velocity of the wave-transmission would be known, from the equations given above, by integration.

46. If, through the partial radiation of heat, or from any other cause, the dynamical equation should take the form

$$\frac{d^2y}{dt^2}=f\left(\frac{dy}{dx}, \ \frac{dy}{dt}\right)\cdot\frac{d^2y}{dx^2},$$

we must integrate it as before by assuming

$$\frac{dy}{dt}=\mathrm{F}\left(\frac{dy}{dx}\right);$$

which gives

$$(\mathrm{F}'\alpha)^2=f(\alpha, \ \mathrm{F}\alpha).$$

This equation being integrated will furnish the form of F; and then the integral of the proposed dynamical equation will be

$$\begin{cases} y=\alpha x+(\mathrm{C}\pm\mathrm{F}\alpha)t+\phi\alpha, \\ x=\mp\mathrm{F}'\alpha.t-\phi'\alpha, \end{cases}$$

which does not present any new difficulty.